

m -KOSZUL ARTIN-SCHELTER REGULAR ALGEBRAS

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ABSTRACT. This paper studies the homological determinants and Nakayama automorphisms of not-necessarily-noetherian m -Koszul twisted Calabi-Yau or, equivalently, m -Koszul Artin-Schelter regular, algebras. Dubois-Violette showed that such an algebra is isomorphic to a derivation quotient algebra $\mathcal{D}(\mathbf{w}, i)$ for a unique-up-to-scalar-multiples twisted superpotential \mathbf{w} . By definition, $\mathcal{D}(\mathbf{w}, i)$ is the quotient of the tensor algebra TV , where $V = \mathcal{D}(\mathbf{w}, i)_1$, by $(\partial^i \mathbf{w})$, the ideal generated by all i^{th} -order left partial derivatives of \mathbf{w} . The restriction map $\sigma \mapsto \sigma|_V$ is used to identify the group of graded algebra automorphisms of $\mathcal{D}(\mathbf{w}, i)$ with a subgroup of $\text{GL}(V)$. We show that the homological determinant of a graded algebra automorphism σ of an m -Koszul Artin-Schelter regular algebra $\mathcal{D}(\mathbf{w}, i)$ is given by the formula $\text{hdet}(\sigma)\mathbf{w} = \sigma^{\otimes(m+i)}(\mathbf{w})$. It follows from this that the homological determinant of the Nakayama automorphism of an m -Koszul Artin-Schelter regular algebra is 1. As an application, we prove that the homological determinant and the usual determinant coincide for most quadratic noetherian Artin-Schelter regular algebras of dimension 3.

1. INTRODUCTION

We fix a field k . All vector spaces will be k -vector spaces. All algebras will be \mathbb{N} -graded k -algebras. Such an algebra, $A = A_0 \oplus A_1 \oplus \cdots$ is **connected** if $A_0 = k$. In that case, $k = A/A_{\geq 1}$ is a graded left A -module concentrated in degree 0.

1.1. Two classes of graded algebras play a central role in non-commutative projective algebraic geometry, namely twisted Calabi-Yau algebras and Artin-Schelter regular algebras. The definitions of these algebras vary a little from one paper to another but in this paper they will be defined in such a way that they are, in fact, the same (see §2.1). The algebras we study need not be noetherian.

This paper concerns the homological determinants and Nakayama automorphisms of m -Koszul twisted Calabi-Yau or, equivalently, m -Koszul Artin-Schelter regular, algebras.

1.1.1. A connected graded algebra A is **m -Koszul** if it is finitely presented and its relations are homogeneous of degree m and $\text{Ext}_A^i(k, k)$ is concentrated in a single degree for all i . 2-Koszul algebras are Koszul algebras in the “classical” sense. m -Koszul algebras were introduced by R. Berger [2].

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1.1.2. A connected graded algebra A is **Artin-Schelter regular** (AS-regular, for short) of dimension d if $\text{gldim}(A) = d < \infty$, and

$$(1-1) \quad \text{Ext}_A^i(k, A) \cong \begin{cases} k(\ell) & \text{if } i = d \\ 0 & \text{if } i \neq d \end{cases}$$

for some $\ell \in \mathbb{Z}$. Commutative Artin-Schelter regular algebras are polynomial rings. The number ℓ in (1-1) is called the **Gorenstein parameter** of A .

1.2. **Notation.** The following notation will be used throughout the paper.

- (1) A denotes an arbitrary connected \mathbb{N} -graded k -algebra.
- (2) S denotes an m -Koszul AS-regular algebra of dimension d and Gorenstein parameter ℓ .
- (3) $V = S_1$ and $R = \ker(V^{\otimes m} \xrightarrow{\text{multiplication}} S_m)$; thus $S \cong TV/(R)$.
- (4) $\text{Aut}(S)$ is the group of graded k -algebra automorphisms of S . Since an automorphism of S is determined by its restriction to S_1 we always view $\text{Aut}(S)$ as a subgroup of $\text{GL}(V)$, $\text{Aut}(S) = \{\sigma \in \text{GL}(V) \mid \sigma^{\otimes m}(R) = R\}$.
- (5) $\text{hdet} : \text{Aut}(S) \rightarrow k^\times$ is the homological determinant (see Theorem 2.4).
- (6) $\epsilon \in \text{Aut}(S)$ is multiplication by $(-1)^i$ on S_i .
- (7) \mathbf{w} , or \mathbf{w}_S when we need to emphasize S , is a basis for the subspace $\bigcap_{s+t=\ell-m} V^{\otimes s} \otimes R \otimes V^{\otimes t}$ of $V^{\otimes \ell}$.
- (8) $E = \text{Ext}_S^*(k, k)$ and $E^i = \text{Ext}_S^i(k, k)$.
- (9) ν is the Nakayama automorphism of S (defined in §2.1).
- (10) μ is the Nakayama automorphism of E (defined in §§4.2 and 4.3).

Under the convention in (4), $\nu \in \text{GL}(V)$. However, if $m \geq 3$, E is not generated by $V^* = \text{Ext}_S^1(k, k)$ so it is not natural to identify $\text{Aut}(E)$ with a subgroup of $\text{GL}(V^*)$. Nevertheless, we can, and do, consider $\mu|_{V^*}$.

Proposition 2.10 explains why the intersection in (7) is 1-dimensional.

1.3. **Results I.** In this section, we adopt the notation in §1.2.

Theorem 1.1. (Theorem 3.2) $\text{Aut}(S) = \{\sigma \in \text{GL}(V) \mid \sigma^{\otimes \ell}(k\mathbf{w}) = k\mathbf{w}\}$.

Theorem 1.2. (Theorem 3.3) If $\sigma \in \text{Aut}(S)$, then $\sigma^{\otimes \ell}(\mathbf{w}) = \text{hdet}(\sigma)\mathbf{w}$.

The following theorem makes [5, Thms. 6.2 and 6.8] less mysterious (cf. [19, Thm. 4.12]).

Theorem 1.3. (Corollary 4.5) S is Calabi-Yau if and only if \mathbf{w} is a $(-1)^{d+1}$ -twisted superpotential in the sense of Definition 2.5.

Conditions (2) and (3) in the next result are closed conditions so (1) says that the homological determinant equals the determinant for “almost all” noetherian quadratic AS-regular algebras of dimension 3.

Theorem 1.4. (Theorem 7.2) Suppose $d = 3$ and $m = 2$, i.e., S is a 3-dimensional Artin-Schelter regular algebra on 3 generators. The following are equivalent:

- (1) There exists $\sigma \in \text{Aut}(S)$ such that $\text{hdet}(\sigma) \neq \det(\sigma)$.
- (2) $c(\mathbf{w}) \in \text{Sym}^3 V$.
- (3) $R \subseteq \text{Sym}^2 V$.

1.4. Results II. We continue to use the notation in §1.2.

Theorem 1.5. (Thm. 4.8) *The Nakayama automorphism of $\text{Ext}_S^*(k, k)$ is $(\epsilon^{d+1}\nu)^!$.¹*

Lu, Mao, and Zhang, have proved a general result showing that the Nakayama automorphisms of suitable algebras A belong to the center of the full automorphism group [9, Theorem 4.2]. Using Theorem 1.5, we give a different proof of the fact that the Nakayama automorphism of S is in the center of $\text{Aut}(S)$ (Corollary 4.9).

The next two results are non-noetherian m -Koszul versions of [14, Thm.5.3] and [13, Thm.5.4(a)]. Our proofs are different and are, perhaps, simpler.

Theorem 1.6. (Theorem 4.11) $\text{hdet}(\nu) = 1$.

Theorem 1.7. (Theorem 5.5) *Let $\sigma \in \text{Aut}(S)$. The Zhang twist, S^σ , is also an m -Koszul AS-regular algebra of Gorenstein parameter ℓ , and its Nakayama automorphism is $\text{hdet}(\sigma)^{-1}\sigma^\ell\nu$.*

The following theorem extends [17, Cor. 9.3] and [4, Prop. 6.5] to the higher dimensional case.

Theorem 1.8. (Theorem 4.13) *Let x_1, \dots, x_n be a basis for $V = S_1$. Let $\mathbf{x}^t = (x_1, \dots, x_n)$. There is a unique $n \times n$ matrix \mathbf{M} with entries in $V^{\otimes \ell-2}$ such that $\mathbf{w} = \mathbf{x}^t \mathbf{M} \mathbf{x}$ and a unique $Q \in \text{GL}_n(k)$ such that $(\mathbf{x}^t \mathbf{M})^t = Q \mathbf{M} \mathbf{x}$. The Nakayama automorphism of S is the unique $\nu \in \text{GL}(V)$ such that $\nu = (-1)^{d+1} Q^{-t} \in \text{GL}(n, k)$ with respect to this basis. In particular, S is Calabi-Yau if and only if $Q = (-1)^{d+1} \text{Id}$.*

1.5. Methods. We continue to use the notation in §1.2.

Almost all our results depend on Theorem 1.9 below which says that S is completely determined by the 1-dimensional subspace kw defined in §1.2(7).

1.5.1. Derivation-quotient algebras. Let V be a vector space and W a subspace of some tensor power, $V^{\otimes p}$ say. We introduce the following notation:

$$\begin{aligned} \partial W &:= \{(\psi \otimes \text{id}^{\otimes p-1})(\mathbf{w}) \mid \psi \in V^*, \mathbf{w} \in W\}, \\ \partial^{i+1} W &:= \partial(\partial^i W) \quad \text{for all } i \geq 0, \\ \mathcal{D}(W, i) &:= TV/(\partial^i W). \end{aligned}$$

The space $\partial^i W$ appears in [6, §4] where it is denoted $W^{(p-i)}$.

Let $\mathbf{w} \in V^{\otimes \ell}$. Following [5], we introduce the notation

$$\mathcal{D}(\mathbf{w}, i) := \mathcal{D}(k\mathbf{w}, i)$$

and call $\mathcal{D}(\mathbf{w}, i)$ the i -th order derivation-quotient algebra of \mathbf{w} .

Theorem 1.9 (Dubois-Violette). [6, Thm. 11] *In the notation of §1.2*

$$S \cong \mathcal{D}(\mathbf{w}, \ell - m).$$

All results in this paper concern automorphisms of m -Koszul AS-regular algebras so the following simple observation plays a central role.

Theorem 1.10. (Theorem 3.2) $\text{Aut}(\mathcal{D}(\mathbf{w}, i)) = \{\sigma \in \text{GL}(V) \mid \sigma^{\otimes \ell}(k\mathbf{w}) = k\mathbf{w}\}.$

¹See §§2.5 and 2.7.1 for the definition of $(\cdot)^!$.

2. PRELIMINARIES

2.1. Calabi-Yau algebras. Let A be a graded k -algebra, A° its opposite algebra, and $A^e = A \otimes A^\circ$. We consider A as a left A^e -module via $(a \otimes b) \cdot c = acb$. If ν is an automorphism of A we denote by ${}_\nu A_1$ the left A^e -module that is A as a graded vector space with action $(a \otimes b) \cdot c = \nu(a)cb$. We say A is **twisted Calabi-Yau of dimension d** if it has a finite-length resolution as a left A^e -module by finitely generated projective A^e -modules and there is an isomorphism

$$(2-1) \quad \text{Ext}_{A^e}^i(A, A^e) \cong \begin{cases} {}_\nu A_1(\ell) & \text{if } i = d \\ 0 & \text{if } i \neq d \end{cases}$$

of graded right A^e -modules for some integer ℓ and some graded k -algebra automorphism ν . We call ν the **Nakayama automorphism** of A . (Some authors call ν^{-1} the Nakayama automorphism; see, e.g., [13].) We say A is **Calabi-Yau** if it is twisted Calabi-Yau and $\nu = \text{id}_A$. If A is twisted Calabi-Yau of dimension d , then it has global dimension d .

When A is Calabi-Yau of dimension d we often say it is d -Calabi-Yau or d -CY.

Proposition 2.1. [16, Prop.3.1(1)] *If A is Artin-Schelter regular, then ${}_A k$ has a finite free resolution, i.e., a finite projective resolution in which every term is a finitely generated free left A -module. In particular, A is finitely presented.*

Corollary 2.2. *If A is Artin-Schelter regular, then there is a finite dimensional graded vector space V and a finite dimensional graded subspace R of the tensor algebra TV such that $A = TV/(R)$.*

Lemma 2.3. [13, Lem.1.2] *A connected graded algebra is twisted Calabi-Yau of dimension d if and only if it is Artin-Schelter regular algebra of dimension d . In that case, the numbers ℓ in (2-1) and (1-1) are the same.*

2.2. The homological determinant. In [8], Jorgensen and Zhang introduced the homological determinant in order to prove for AS-Gorenstein rings, which are defined in Theorem 2.4, non-commutative analogues of several important results about commutative connected graded Gorenstein algebras. Since then the homological determinant has played an important role in the invariant theory of non-commutative rings.

The homological determinant of an AS-Gorenstein algebra, A , is a group homomorphism $\text{hdet} : \text{Aut}(A) \rightarrow k^\times$. If SV is a symmetric algebra, i.e., a polynomial ring, on a finite dimensional vector space V , then $\text{Aut}(SV) = \text{GL}(V)$ and the homological determinant is the same as usual determinant.

It is often difficult to compute the homological determinant.

The homological determinant is the function hdet in the next theorem.

Theorem 2.4. [8, Lem. 2.2, Prop. 2.5] *Let A be a connected graded k -algebra. Suppose there is an integer d such that left and right injective dimensions of A as a graded A -module equal d and that*

$$\text{Ext}_A^i(k, A) = \text{Ext}_{A^\circ}^i(k, A) \cong \delta_{id} k(\ell).$$

for some integer ℓ . (Such an A is said to be AS-Gorenstein.)

- (1) *Each $\sigma \in \text{Aut}(A)$ induces an isomorphism $H_{\mathfrak{m}}^d(\sigma) : H_{\mathfrak{m}}^d(A) \rightarrow H_{\mathfrak{m}}^d(A)$ of graded vector spaces where $H_{\mathfrak{m}}^d(A)$ is the d^{th} local cohomology group (see [8] for the definition.)*

- (2) There is an isomorphism $\psi : H_m^d(A) \rightarrow A'(\ell)$ of graded vector spaces where A' is the Matlis dual, $A' = \bigoplus_{i=0}^{\infty} \text{Hom}_k(A_i, k)$.
 (3) There is a group homomorphism

$$\text{hdet} : \text{Aut}(A) \rightarrow k^\times, \quad \sigma \mapsto \text{hdet}(\sigma),$$

such that the diagram

$$\begin{array}{ccc} H_m^d(A) & \xrightarrow{H_m^d(\sigma)} & H_m^d(A) \\ \psi \downarrow & & \downarrow \psi \\ A'(\ell) & \xrightarrow{\text{hdet}(\sigma) (\sigma^{-1})'} & A'(\ell) \end{array}$$

commutes.

- (4) The maps $H_m^d(\sigma)$ and $(\sigma^{-1})'$ are σ -linear: a degree-preserving linear map $f : M \rightarrow N$ between two graded left A -modules is σ -linear if $f(am) = \sigma(a)f(m)$ for all $a \in A$ and $m \in M$.

2.3. Let $\phi : V^{\otimes \ell} \rightarrow V^{\otimes \ell}$ be the linear map

$$(2-2) \quad \phi(v_1 \otimes v_2 \otimes \cdots \otimes v_{\ell-1} \otimes v_\ell) := v_\ell \otimes v_1 \otimes \cdots \otimes v_{\ell-2} \otimes v_{\ell-1}.$$

Definition 2.5. Let $\mathbf{w} \in V^{\otimes \ell}$ and $\sigma \in \text{GL}(V)$. We call \mathbf{w} a

- (1) **superpotential** if $\phi(\mathbf{w}) = \mathbf{w}$;
- (2) **σ -twisted superpotential** if $(\sigma \otimes \text{id}^{\otimes \ell-1})\phi(\mathbf{w}) = \mathbf{w}$;
- (3) **a twisted superpotential** if it is a σ -twisted superpotential for some σ .

Remark 2.6. Our terminology does not agree with that in [5]: there, $\mathbf{w} \in V^{\otimes \ell}$ is called a twisted superpotential if it is invariant under the map $v_1 \otimes \cdots \otimes v_\ell \mapsto (-1)^{\ell+1} \sigma(v_\ell) \otimes v_1 \otimes \cdots \otimes v_{\ell-1}$ for some $\sigma \in \text{GL}(V)$.

2.4. For a linear transformation $\theta : U \rightarrow V$, we define $\theta^* : V^* \rightarrow U^*$ by $\theta^*(\xi) = \xi \circ \theta$. In order to identify $(V^*)^{\otimes i}$ with $(V^{\otimes i})^*$, we adopt the convention that

$$(\xi_1 \otimes \cdots \otimes \xi_i)(v_1 \otimes \cdots \otimes v_i) := \xi_1(v_1) \cdots \xi_i(v_i).$$

2.5. The m -Koszul complex. The m -homogeneous dual of an m -homogeneous algebra $A = TV/(R)$ is the algebra

$$A^! := \frac{TV^*}{(R^\perp)}$$

where $R^\perp \subseteq (V^{\otimes m})^* = (V^*)^{\otimes m}$ consists of the functions vanishing on R .

Since $\text{Aut}(A) = \{\sigma \in \text{GL}(V) \mid \sigma^{\otimes m}(R) = R\}$, the anti-isomorphism $\text{GL}(V) \rightarrow \text{GL}(V^*)$, $\sigma \mapsto \sigma^*$, restricts to an anti-isomorphism

$$\text{Aut}(A) \rightarrow \text{Aut}(A^!)$$

that we denote by $\sigma \mapsto \sigma^!$.

Lemma 2.7. Let $A = TV/(R)$ be an m -homogeneous algebra. For all $i \geq 0$, define

$$(2-3) \quad W_i := \begin{cases} V^{\otimes i} & \text{if } 0 \leq i \leq m-1, \\ \bigcap_{s+m+t=i} V^{\otimes s} \otimes R \otimes V^{\otimes t} \subseteq V^{\otimes i} & \text{if } i \geq m. \end{cases}$$

The quotient map $\Phi : TV^* \rightarrow A^!$ induces isomorphisms $A_i^! \cong W_i^*$ for all $i \geq 0$.

Proof. Let Φ_i denote the restriction of Φ to $(V^*)^{\otimes i} = (V^{\otimes i})^*$. There is an exact sequence $0 \rightarrow W_i^\perp \rightarrow (V^*)^{\otimes i} \rightarrow W_i^* \rightarrow 0$ so it suffices to show that $\ker(\Phi_i) = W_i^\perp$.

Suppose $0 \leq i \leq m-1$. Then $A_i^! = (V^*)^{\otimes i}$ so $\ker(\Phi_i) = 0$. But $W_i = V^{\otimes i}$ so $W_i^\perp = 0$ also. Hence $\ker(\Phi_i) = W_i^\perp$ when $i \leq m-1$.

Suppose $i \geq m$. Then

$$\begin{aligned} W_i^\perp &= \left(\bigcap_{s+m+t=i} V^{\otimes s} \otimes R \otimes V^{\otimes t} \right)^\perp \\ &= \sum_{s+m+t=i} (V^{\otimes s} \otimes R \otimes V^{\otimes t})^\perp \\ &= \sum_{s+m+t=i} (V^*)^{\otimes s} \otimes R^\perp \otimes (V^*)^{\otimes t} \\ &= \ker(\Phi_i). \end{aligned}$$

The proof is complete. \square

The space W_i is denoted by $W^{(i)}$ in [6] and by J_i in [2].

Let $\{x_i\}$ be a basis for V and $\{\xi_i\}$ the dual basis for $V^* = A_1^!$. The element $e := \sum_i x_i \otimes \xi_i$ belongs to $A \otimes A^!$ and is independent of the choice of basis for V . Right multiplication by e is a left A -module homomorphism

$$A \otimes (A_n^!)^* \xrightarrow{\cdot e} A \otimes (A_{n-1}^!)^*.$$

When A is an m -homogeneous algebra $e^m = 0$.

Theorem 2.8 (Berger). [4, Thm. 2.4] *Let A be an m -homogeneous algebra and define $P_i = A \otimes (A_i^!)^*$ for $i \geq 0$. The complex*

$$\cdots \xrightarrow{\cdot e} P_{2m} \xrightarrow{\cdot e^{m-1}} P_{m+1} \xrightarrow{\cdot e} P_m \xrightarrow{\cdot e^{m-1}} P_1 \xrightarrow{\cdot e} P_0 \longrightarrow k \longrightarrow 0$$

is exact if and only if A is m -Koszul.

If A is an m -homogeneous algebra we call the complex in Theorem 2.8 the m -Koszul complex for A .

2.6. The element w in §1.2(7). The definition of an AS-regular algebra A implies that if P_\bullet is a deleted projective resolution of ${}_A k$, then $\text{Hom}_A(P_\bullet, A)$ is a deleted projective resolution of k_A . This implies the following well-known result.

Proposition 2.9. *With the notation in §1.2, let $P_i = S \otimes (S_i^!)^*$ for $i \geq 0$. The left-hand part of the minimal resolution of ${}_S k$ looks like*

$$0 \longrightarrow P_\ell \xrightarrow{\cdot e} P_{\ell-1} \xrightarrow{\cdot e^{m-1}} P_{\ell-m} \xrightarrow{\cdot e} \cdots$$

where $\dim_k(S_\ell^!) = 1$, $\dim_k(S_{\ell-1}^!) = \dim_k(V)$, and $\dim_k(S_{\ell-m}^!) = \dim_k(R)$.

2.6.1. Remark. With the notation in §1.2, the fact that the arrow $P_\ell \rightarrow P_{\ell-1}$ is multiplication by e implies that d is odd if $m \geq 3$.

2.6.2. The intersection in the next proposition is W_ℓ which is isomorphic to $(S_\ell^!)^*$.

Proposition 2.10. *With the notation in §1.2, $\bigcap_{s+t=\ell-m} V^{\otimes s} \otimes R \otimes V^{\otimes t}$ is a 1-dimensional subspace of $V^{\otimes \ell}$.*

We retain the notation in §1.2. Dubois-Violette's theorem (Theorem 1.9 above) says that S is isomorphic to $\mathcal{D}(\mathbf{w}', \ell - m)$ for *some* twisted superpotential $\mathbf{w}' \in V^{\otimes \ell}$. Proposition 2.12 below shows that we can take \mathbf{w}' to be the element \mathbf{w} defined in §1.2(7). This result is implicit in [6, Thm. 11] however the following proof is short and explicit (cf. [19, Thm.4.12]).

Lemma 2.11. *Suppose m, p, q are integers ≥ 0 . Let R be a subspace of $V^{\otimes m}$ and $\mathbf{w} \in V^{\otimes p} \otimes R \otimes V^{\otimes q}$. If \mathbf{w} is a twisted superpotential, then*

$$\mathbf{w} \in \bigcap_{s+t=p+q} V^{\otimes s} \otimes R \otimes V^{\otimes t},$$

where the intersection is taken over all integers $s, t \geq 0$ such that $s + t = p + q$.

Proof. Let $\sigma \in \text{GL}(V)$ be such that $(\sigma \otimes \text{id})\phi(\mathbf{w}) = \mathbf{w}$. Then

$$\begin{aligned} \mathbf{w} &= (\sigma \otimes \text{id})\phi(\mathbf{w}) \in V^{\otimes p+1} \otimes R \otimes V^{\otimes q-1} & \text{if } q \geq 1, \text{ and} \\ \mathbf{w} &= \phi^{-1}(\sigma^{-1} \otimes \text{id})\mathbf{w} \in V^{\otimes p-1} \otimes R \otimes V^{\otimes q+1} & \text{if } p \geq 1. \end{aligned}$$

An induction argument completes the proof. \square

Proposition 2.12. *Let $\ell \geq m \geq 2$. Let $\mathbf{w} \in V^{\otimes \ell}$. Suppose $\mathcal{D}(\mathbf{w}, \ell - m)$ is an m -Koszul AS-regular algebra of Gorenstein parameter ℓ . Let $R = \partial^{\ell-m}(k\mathbf{w})$. If \mathbf{w} is a twisted superpotential, then*

$$k\mathbf{w} = \bigcap_{s+t+m=\ell} V^{\otimes s} \otimes R \otimes V^{\otimes t},$$

where the intersection is taken over all integers $s, t \geq 0$ such that $s + m + t = \ell$.

Proof. By definition, $\mathcal{D}(\mathbf{w}, \ell - m) = TV/(R)$. It follows from the definition of $\partial^{\ell-m}$ that $\mathbf{w} \in V^{\otimes \ell-m} \otimes R$. Since \mathbf{w} is a twisted superpotential, Lemma 2.11 tells us that $\mathbf{w} \in W_\ell$. But $\dim_k(W_\ell) = 1$. \square

2.7. The algebra structure on $\text{Ext}_S^*(k, k)$. The Yoneda Ext-algebra of a connected graded k -algebra A is

$$E(A) := \bigoplus_{i=0}^{\infty} \text{Ext}_A^i(k, k).$$

Because A is a graded k -algebra, $E(A)$ is bi-graded. Several notations are used for its homogeneous components, namely $E^{ij}(A) = E^i(A)_{-j} = \text{Ext}_A^i(k, k)_{-j}$.

We denote the product on $A^!$ by $f \cdot g$.

Following [7, §2], we define

$$\rho(i) := \begin{cases} jm & \text{if } i = 2j \\ jm + 1 & \text{if } i = 2j + 1 \end{cases}$$

and the bigraded vector space

$$\mathcal{E} = \mathcal{E}(A) = \bigoplus_j \mathcal{E}_j^i \quad \text{where} \quad \mathcal{E}_j^i = \begin{cases} 0 & \text{if } j \neq -\rho(i) \\ A_{\rho(i)}^! & \text{if } j = -\rho(i). \end{cases}$$

We define a multiplication $*$ on \mathcal{E} as follows: if $f \in \mathcal{E}^i$ and $g \in \mathcal{E}^j$, then

$$(2-4) \quad f * g = \begin{cases} (-1)^{ij} f \cdot g & \text{if } m = 2 \\ f \cdot g & \text{if } m > 2 \text{ and at least one of } i \text{ and } j \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.13. [4, Prop. 3.1], [7, Prop. 2.3]. *Let A be an m -Koszul algebra. The direct sum of the vector space isomorphisms $A_{\rho(i)}^! \rightarrow \text{Ext}_A^i(k, k)$ is an isomorphism of bigraded k -algebras,*

$$\mathcal{E}_*^* \xrightarrow{\sim} \text{Ext}_A^*(k, k)_*.$$

When A is an m -Koszul algebra we identify \mathcal{E} with $\text{Ext}_A^*(k, k)$.

2.7.1. *The automorphisms $\sigma^!$ of \mathcal{E} .* It follows from the definition of multiplication on \mathcal{E} that every graded k -algebra automorphism of $A^!$ “restricts” to a graded k -algebra automorphism of \mathcal{E} . Thus, if $\sigma \in \text{Aut}(A)$ and $\sigma^!$ is the automorphism of $A^!$ induced by σ^* , then $\sigma^!$ “restricts” to an automorphism of \mathcal{E} . We continue to denote that automorphism of \mathcal{E} by $\sigma^!$ (cf., [19, p.57]).

Let $\sigma \in \text{Aut}(A)$. By Lemma 2.7, $\text{Ext}_A^i(k, k)$ is a quotient of $(V^*)^{\otimes i}$ so the action of $\sigma^!$ on $\text{Ext}_A^i(k, k)$ is induced by the action of $(\sigma^!)^{\otimes i}$ on $(V^*)^{\otimes i}$. In particular, that is the meaning of $\sigma^!$ in (2-5).²

Proposition 2.14. [19, Prop. 1.11] *With the notation in §1.2, the homological determinant of $\sigma \in \text{Aut}(S)$ has the property that*

$$(2-5) \quad \text{hdet}(\sigma)u = \sigma^!(u)$$

for all $u \in \text{Ext}_S^d(k, k)$.

3. HOMOLOGICAL DETERMINANTS

Theorem 3.3 gives a simple formula for the homological determinant of an m -Koszul AS-regular algebra. This formula is the key to several results in later sections.

Lemma 3.1. *Let $\sigma \in \text{GL}(V)$ and $W \subseteq V^{\otimes j}$. If $\sigma^{\otimes j}(W) = W$, then $\sigma \in \text{Aut}(\mathcal{D}(W, i))$ for all i .*

Proof. To prove the result it suffices to show that $\sigma^{\otimes j-i}(\partial^i W) = \partial^i W$ for all $i = 0, \dots, j$. First we prove the claim for $i = 1$. It is clear that

$$\begin{aligned} \sigma^{\otimes j-1}(\partial W) &= \{\sigma^{\otimes j-1}(\psi \otimes \text{id}^{\otimes j-1})(v) \mid \psi \in V^*, v \in W\} \\ &= \{(\psi \otimes \sigma^{\otimes j-1})(v) \mid \psi \in V^*, v \in W\} \\ &= \{(\psi \sigma \otimes \sigma^{\otimes j-1})(v) \mid \psi \in V^*, v \in W\} \\ &= \{(\psi \otimes \text{id}^{\otimes j-1})(\sigma \otimes \sigma^{\otimes j-1})(v) \mid \psi \in V^*, v \in W\} \\ &= \partial(\sigma^{\otimes j}(W)) \\ &= \partial W. \end{aligned}$$

²The convention we use in §2.4 to identify $(V^*)^{\otimes i}$ with $(V^{\otimes i})^*$ differs from that in [19]. The paper [19] uses the convention $(\xi_1 \otimes \dots \otimes \xi_i)(v_1 \otimes \dots \otimes v_i) = \xi_i(v_1) \dots \xi_1(v_i)$. Under that convention, if $\sigma_1, \dots, \sigma_i \in \text{GL}(V)$ and $\xi_1, \dots, \xi_i \in V^*$, then $(\sigma_1 \otimes \dots \otimes \sigma_i)^*(\xi_1 \otimes \dots \otimes \xi_i) = \sigma_i^*(\xi_1) \dots \sigma_1^*(\xi_i)$. However, if $\sigma \in \text{GL}(V)$ both conventions give $(\sigma \otimes \dots \otimes \sigma)^*(\xi_1 \otimes \dots \otimes \xi_i) = \sigma^*(\xi_1) \dots \sigma^*(\xi_i)$. Therefore Proposition 2.14 holds as stated under both conventions.

It now follows by induction that $\sigma^{\otimes j-i}(\partial^i W) = \partial^i W$ for all i .

By definition, $\mathcal{D}(W, i) = TV/(\partial^i W)$ so σ is an automorphism of $\mathcal{D}(W, i)$. \square

Theorem 3.2. *With the notation in §1.2,*

$$\text{Aut}(S) = \{\sigma \in \text{GL}(V) \mid \sigma^{\otimes \ell}(kw) = kw\}.$$

Proof. Recall that $V = S_1$. Let $\sigma \in \text{GL}(V)$.

Suppose $\sigma \in \text{Aut}(S)$. By definition, $R = \ker(V^{\otimes m} \xrightarrow{\text{mult}} S_m)$, so $\sigma^{\otimes m}(R) = R$. Therefore $\sigma^{\otimes i}(V^{\otimes s} \otimes R \otimes V^{\otimes t}) = V^{\otimes s} \otimes R \otimes V^{\otimes t}$ for all s and t such that $s+m+t = i$. Hence $\sigma^{\otimes i}(W_i) = W_i$ for all i . But $W_\ell = kw$ so $\sigma^{\otimes \ell}(kw) = kw$.

Conversely, suppose $\sigma^{\otimes \ell}(kw) = kw$. By Proposition 2.12, $S = \mathcal{D}(kw, \ell - m)$, so $\sigma \in \text{Aut}(S)$ by Lemma 3.1. \square

Theorem 3.3. *Adopt the notation in §1.2. If $\sigma \in \text{Aut}(S)$, then*

$$\sigma^{\otimes \ell}(w) = \text{hdet}(\sigma)w.$$

Proof. By Theorem 3.2, $\sigma^{\otimes \ell}(w) = \lambda w$ for some $0 \neq \lambda \in k$. Hence $\sigma^{\otimes \ell}$ is multiplication by λ on $W_\ell = kw$. Therefore, if $u \in W_\ell^*$, then $(\sigma^*)^{\otimes \ell}(u) = (\sigma^{\otimes \ell})^*(u) = \lambda u$. But $W_\ell^* \cong \text{Ext}_S^d(k, k)$ and, by Proposition 2.14, $\text{hdet}(\sigma)u = (\sigma^*)^{\otimes \ell}(u)$ for all $u \in \text{Ext}_S^d(k, k)$. Therefore $\lambda = \text{hdet}(\sigma)$. \square

4. NAKAYAMA AUTOMORPHISMS

We continue to use the notation and assumptions in §1.2. This section proves several results about the Nakayama automorphisms of S and $\text{Ext}_S^*(k, k)$.

4.1. The term “Nakayama automorphism” has different meanings in the literature. We defined the term for Calabi-Yau algebras in §2.1 but there is an older usage.

A Frobenius algebra is a finite dimensional k -algebra R such that $R \cong R^*$ as left R -modules. Let $\alpha : R \rightarrow R^*$ be a left R -module isomorphism. The bilinear form $\langle \cdot, \cdot \rangle : R \times R \rightarrow k$ defined by $\langle x, y \rangle = \alpha(y)(x)$ has the property that $\langle a, bc \rangle = \langle ab, c \rangle$ and, because α is injective, $\langle \cdot, \cdot \rangle$ is non-degenerate. Because $\langle \cdot, \cdot \rangle$ is non-degenerate there is a unique linear map $\mu : R \rightarrow R$ such that $\langle a, b \rangle = \langle b, \mu(a) \rangle$ for all $a, b \in R$. The map μ is called a Nakayama automorphism of R . Because $\langle a, bc \rangle = \langle ab, c \rangle$, μ is an algebra automorphism. Of course, μ depends on the choice of isomorphism α .

4.2. A finite dimensional \mathbb{N} -graded k -algebra A is a **graded Frobenius algebra** if there is an integer ℓ such that $A \cong A^*(-\ell)$ as graded left A -modules where $(A^*)_i = (A_{-i})^*$. Let A be a connected graded Frobenius algebra and let A_ℓ be its top-degree non-zero component. Then $\dim_k(A_\ell) = 1$. Let v be a basis for A_ℓ . There is, up to non-zero scalar multiples, a unique isomorphism $A \rightarrow A^*(-\ell)$ of graded left A -modules and hence a unique (up to non-zero scalar multiples) non-degenerate degree-preserving bilinear map $\langle \cdot, \cdot \rangle : A \times A \rightarrow k(-\ell)$ such that $\langle ab, c \rangle = \langle a, bc \rangle$ for all $a, b, c \in A$, namely

$$\langle x, y \rangle = \begin{cases} \lambda & \text{if } x \in A_i \text{ and } y \in A_{\ell-i} \text{ and } xy = \lambda v \\ 0 & \text{otherwise.} \end{cases}$$

Although $\langle \cdot, \cdot \rangle$ depends on the choice of v , the linear map $\mu : A \rightarrow A$ such that $\langle x, y \rangle = \langle y, \mu(x) \rangle$ for all $x, y \in A$ does not. We call μ *the* Nakayama automorphism of A .

Since $\langle 1, \mathbf{v} \rangle = \langle \mathbf{v}, 1 \rangle$, $\mu(\mathbf{v}) = \mathbf{v}$.

Let x be a non-zero element in A_1 . Then there is $y \in A_{\ell-1}$ such that $\langle x, y \rangle = 1$, i.e., $xy = \mathbf{v}$. But $\langle x, y \rangle = \langle y, \mu(x) \rangle$ so $\mathbf{v} = xy = y\mu(x)$

Proposition 4.1. *Let A be a connected graded Frobenius algebra. Its Nakayama automorphism belongs to the center of $\text{Aut}(A)$.*

Proof. The proof uses some of the observations in [15, §3].

Let μ be the Nakayama automorphism and $\sigma \in \text{Aut}(A)$. Let A_n be the top-degree component of A and \mathbf{v} a basis for A_n . The bilinear form $\langle \cdot, \cdot \rangle$ on A is, up to non-zero scalar multiple, given by the formula $\langle x, y \rangle \mathbf{v} = xy$ for $x \in A_i$ and $y \in A_{n-i}$.

Let $x \in A_i$ and $y \in A_{n-i}$. Since $xy = y\mu(x)$,

$$\sigma(y)\mu\sigma(x) = \sigma(x)\sigma(y) = \sigma(xy) = \sigma(y\mu(x)) = \sigma(y)\sigma\mu(x),$$

$\langle \sigma(y), \mu\sigma(x) \rangle = \langle \sigma(y), \sigma\mu(x) \rangle$. Since $\langle \cdot, \cdot \rangle : A_{n-i} \times A_i \rightarrow k$ is non-degenerate, $\mu\sigma(x) = \sigma\mu(x)$. Thus $\mu\sigma = \sigma\mu$. \square

4.3. By [4, Cor. 5.12], E is a graded Frobenius algebra.³

After the discussion in §§2.7 and 4.1, the Frobenius pairing $\langle \cdot, \cdot \rangle : E \times E \rightarrow k$ is given by the formula

$$\langle \xi, \eta \rangle = (-1)^{i(d-i)}(\xi\eta)(\mathbf{w}) \quad \text{for } \xi \in E^i \text{ and } \eta \in E^{d-i}$$

and the Nakayama automorphism is the unique linear map $\mu : E \rightarrow E$ such that

$$(4-1) \quad \langle \xi, \eta \rangle = \langle \eta, \mu(\xi) \rangle$$

for all $\xi \in E^i$ and $\eta \in E^{d-i}$ and all i (cf. [4, p.91] which uses the notation (\cdot, \cdot) where we use $\langle \cdot, \cdot \rangle$, and they use $\langle \cdot, \cdot \rangle$ to denote the pairing between a vector space and its linear dual).

Lemma 4.2. *We adopt the notation and hypotheses in §1.2. If $\sigma \in \text{Aut}(S)$, then*

$$\langle \sigma^1(\xi), \sigma^1(\eta) \rangle = \text{hdet}(\sigma)\langle \xi, \eta \rangle$$

for all $\xi \in E^i$, all $\eta \in E^{d-i}$, and all $i = 0, \dots, d$.

Proof. The calculation

$$\begin{aligned} \langle \sigma^1(\xi), \sigma^1(\eta) \rangle &= (-1)^{i(d-i)}(\sigma^1(\xi)\sigma^1(\eta))(\mathbf{w}) \\ &= (-1)^{i(d-i)}(\sigma^1(\xi\eta))(\mathbf{w}) \\ &= (-1)^{i(d-i)}(\xi\eta)(\sigma^{\otimes \ell}(\mathbf{w})) \\ &= (-1)^{i(d-i)}(\xi\eta)(\text{hdet}(\sigma)\mathbf{w}) \quad (\text{by Theorem 3.3}) \\ &= \text{hdet}(\sigma)(-1)^{i(d-i)}(\xi\eta)(\mathbf{w}) \\ &= \text{hdet}(\sigma)\langle \xi, \eta \rangle \end{aligned}$$

proves the lemma. \square

The next result follows from [4, Thm. 6.3].

Lemma 4.3. *With the notation in §1.2, $\mu|_{V^*} = (-1)^{d+1}\nu^*$.*

³This is a consequence of a more general result in [10] for A_∞ -algebras but the proof in [4] is more elementary.

Theorem 4.4. *We adopt the notation in §1.2. The Nakayama automorphism of S is the unique $\nu \in \text{GL}(V)$ such that*

$$(\nu \otimes \text{id}^{\otimes \ell-1})\phi(\mathbf{w}) = (-1)^{d+1}\mathbf{w}.$$

In particular, \mathbf{w} is a $(-1)^{d+1}\nu$ -twisted superpotential.

Proof. Let ν be the Nakayama automorphism of S . If $\xi \in E^1$ and $\eta \in E^{d-1}$, then

$$\begin{aligned} (\xi\eta)(\mathbf{w}) &= (-1)^{d-1}\langle \xi, \eta \rangle \\ &= (-1)^{d-1}\langle \eta, \mu(\xi) \rangle \\ &= (\eta\mu(\xi))(\mathbf{w}) \\ &= (\mu(\xi)\eta)(\phi(\mathbf{w})) \\ &= ((\mu \otimes \text{id}^{\otimes \ell-1})(\xi\eta))(\phi(\mathbf{w})). \end{aligned}$$

By Lemma 4.3, $(\mu|_{V^*})^* = (-1)^{d+1}\nu$ so

$$(\mu \otimes \text{id}^{\otimes \ell-1})(\xi\eta) = (\xi\eta) \circ ((-1)^{d+1}\nu \otimes \text{id}^{\otimes \ell-1}).$$

Therefore $(\nu \otimes \text{id}^{\otimes \ell-1})(\phi(\mathbf{w})) = (-1)^{d+1}\mathbf{w}$.

Conversely, suppose that $\nu' \in \text{Aut}(S) \subset \text{GL}(V)$ satisfies $(\nu' \otimes \text{id}^{\otimes \ell-1})\phi(\mathbf{w}) = (-1)^{d+1}\mathbf{w}$. Let $\mu' := (-1)^{d+1}(\nu')^*$. If $\xi \in E^1$ and $\eta \in E^{d-1}$, then

$$\begin{aligned} \langle \eta, \mu'(\xi) \rangle &= (-1)^{d-1}(\eta\mu'(\xi))(\mathbf{w}) \\ &= (-1)^{d-1}(\mu'(\xi)\eta)(\phi(\mathbf{w})) \\ &= (-1)^{d-1}((\mu' \otimes \text{id}^{\otimes \ell-1})(\xi\eta))(\phi(\mathbf{w})) \\ &= (-1)^{d-1}(\xi\eta)((-1)^{d+1}\nu' \otimes \text{id}^{\otimes \ell-1})(\phi(\mathbf{w})) \\ &= (-1)^{d-1}(\xi\eta)(\mathbf{w}) \\ &= \langle \xi, \eta \rangle. \end{aligned}$$

But $\langle \xi, \eta \rangle = \langle \eta, \mu(\xi) \rangle$ so $\mu' = \mu|_{V^*}$. However, by Lemma 4.3, $\mu|_{V^*} = (-1)^{d+1}\nu^*$ so

$$(-1)^{d+1}(\nu')^* = \mu' = \mu|_{V^*} = (-1)^{d+1}\nu^*.$$

Thus, as elements of $\text{GL}(V)$, and hence as elements in $\text{Aut}(S)$, $\nu' = \nu$. \square

Corollary 4.5. *With the notation in §1.2, S is Calabi-Yau if and only if $\phi(\mathbf{w}) = (-1)^{d+1}\mathbf{w}$.*

Remark 4.6. Let S be an m -Koszul AS-regular algebra of Gorenstein parameter ℓ . For $m = 2$, S is Calabi-Yau if and only if $\phi(\mathbf{w}) = (-1)^{\ell+1}\mathbf{w}$ (a superpotential in the sense of [5]) by [5, Thm. 6.2], and, for $m > 2$, S is Calabi-Yau if and only if $\phi(\mathbf{w}) = \mathbf{w}$ (a superpotential in our sense) by [5, Thm. 6.8]. (If $m \geq 3$, then d is odd.) The above corollary combines these results and makes them less mysterious. (Note that one direction of the above corollary was essentially proved in [19, Thm. 4.12].)

Example 4.7. Let V be a vector space with basis $\{x, y\}$. Let

$$\mathbf{w} = x^2y^2 + yx^2y + y^2x^2 + xy^2x \in V^{\otimes 4}.$$

The algebra

$$S = \mathcal{D}(\mathbf{w}, 1) = \frac{k\langle x, y \rangle}{(xy^2 + y^2x, x^2y + yx^2)}$$

is a 3-Koszul AS-regular algebra of dimension 3 with Gorenstein parameter 4. By [13, Example 1.6], S is Calabi-Yau. Since $\phi(\mathbf{w}) = \mathbf{w}$, \mathbf{w} is a superpotential in our sense, but not a superpotential in the sense of [5].

4.4. Let S be an m -Koszul AS-regular algebra and $\sigma \in \text{Aut}(S)$. The next result gives a formula for the Nakayama automorphism μ of E in terms of that for S , which we denote by ν . When $m = 2$, i.e., when S is a Koszul algebra, then E is generated in degree 1 so μ is determined by its restriction to $E_1 = V^*$ so, by Lemma 4.3 ([4, Thm. 6.3]), $(-1)^{d+1}\nu^*$. When $m \geq 3$, E is not generated in degree 1, so μ is not determined by its restriction to E_1 . Hence the need for the somewhat awkward calculations in the next proof.

Theorem 4.8. *With the notation in §1.2, $\mu = (\epsilon^{d+1}\nu)^!$.*

Proof. Let $\xi \in E^i$ and $\eta \in E^{d-i}$. Then

$$\begin{aligned} (-1)^{i(d-i)} \langle \eta, (\epsilon^{d+1}\nu)^!(\xi) \rangle &= [\eta(\epsilon^{d+1}\nu)^!(\xi)](\mathbf{w}) \\ &= [(\epsilon^{d+1}\nu)^!(\xi)\eta](\phi^{\rho(i)}(\mathbf{w})) \\ &= (-1)^{(d+1)\rho(i)} [\nu^!(\xi)\eta](\phi^{\rho(i)}(\mathbf{w})). \end{aligned}$$

Repeated use of Theorem 4.4 and an induction argument on j shows that

$$\phi^j(\mathbf{w}) = (-1)^{(d+1)j} ((\nu^{-1})^{\otimes i} \otimes \text{id}^{\otimes \ell-j})(\mathbf{w})$$

for $0 \leq j \leq \ell$. In particular,

$$\phi^{\rho(i)}(\mathbf{w}) = (-1)^{(d+1)\rho(i)} ((\nu^{-1})^{\otimes \rho(i)} \otimes \text{id}^{\otimes \ell-\rho(i)})(\mathbf{w})$$

so

$$\begin{aligned} (-1)^{i(d-i)} \langle \eta, (\epsilon^{d+1}\nu)^!(\xi) \rangle &= (-1)^{(d+1)\rho(i)} [\nu^!(\xi)\eta](\phi^{\rho(i)}(\mathbf{w})) \\ &= [\nu^!(\xi)\eta] \left(((\nu^{-1})^{\otimes \rho(i)} \otimes \text{id}^{\otimes \ell-\rho(i)})(\mathbf{w}) \right) \end{aligned}$$

which equals

$$\left[((\nu^*)^{\otimes \rho(i)} \otimes \text{id}^{\otimes \ell-\rho(i)})(\xi\eta) \right] \left(((\nu^{-1})^{\otimes \rho(i)} \otimes \text{id}^{\otimes \ell-\rho(i)})(\mathbf{w}) \right).$$

This, in turn, is equal to

$$(\xi\eta) \left([\nu^{\otimes \rho(i)} \otimes \text{id}^{\otimes \ell-\rho(i)}] [(\nu^{-1})^{\otimes \rho(i)} \otimes \text{id}^{\otimes \ell-\rho(i)}](\mathbf{w}) \right).$$

But this is equal to $(\xi\eta)(\mathbf{w})$ which is $(-1)^{i(d-i)} \langle \xi, \eta \rangle$. Thus, we have shown that

$$(-1)^{i(d-i)} \langle \eta, (\epsilon^{d+1}\nu)^!(\xi) \rangle = (-1)^{i(d-i)} \langle \xi, \eta \rangle$$

from which it follows that $(\epsilon^{d+1}\nu)^!$ is the Nakayama automorphism of E . \square

A more general case of the following theorem was proved in [9, Thm.4.2] using Hochschild cohomology (cf. [13, Thm.3.11] for the noetherian case). The following is a different proof using Proposition 4.1.

Corollary 4.9. *The Nakayama automorphism of an m -Koszul AS-regular algebra S belongs to the center of $\text{Aut}(S)$.*

Proof. Let $\sigma \in \text{Aut}(S)$. Since $(\sigma\nu)^! = \nu^!\sigma^!$ and μ commutes with $\sigma^!$,

$$(\epsilon^{d+1}\sigma\nu)^! = (\epsilon^{d+1}\nu)^!\sigma^! = \mu\sigma^! = \sigma^!\mu = \sigma^!(\epsilon^{d+1}\nu)^! = (\epsilon^{d+1}\nu\sigma)^!.$$

Hence $\sigma\nu = \nu\sigma$ for all $\sigma \in \text{Aut}(S)$. \square

The next lemma follows from [4, Prop. 5.3].

Lemma 4.10. *Adopt the notation in §1.2. If $m = 2$, then S is Koszul in the “classical” sense and $\ell = d$. If $m \geq 3$, then d is odd and $\ell = \frac{1}{2}m(d-1) + 1$.*

4.5. Suppose for a moment that ν is the Nakayama automorphism of a noetherian connected graded AS-regular algebra. Reyes-Rogalski-Zhang have proved that $\text{hdet}(\nu) = 1$ [14, Thm. 5.3]. This improved their earlier result which required A to also be Koszul [13, Thm.6.3]. The next result extends this to the m -Koszul case without the noetherian hypothesis.

Theorem 4.11. *With the notation in §1.2, $\text{hdet}(\nu) = 1$.*

Proof. Let $\xi \in E^i$ and $\eta \in E^{d-i}$. By Theorem 4.8, $\mu = (\epsilon^{d+1}\nu)^!$ so

$$\begin{aligned} \langle \xi, \eta \rangle &= \langle \eta, \mu(\xi) \rangle = \langle \mu(\xi), \mu(\eta) \rangle \\ &= \langle (\epsilon^{d+1}\nu)^!(\xi), (\epsilon^{d+1}\nu)^!(\eta) \rangle \\ &= \text{hdet}(\epsilon^{d+1}\nu) \langle \xi, \eta \rangle \end{aligned}$$

where the last equality is given by Lemma 4.2. Hence $\text{hdet}(\epsilon^{d+1}\nu) = 1$. Now, by Theorem 3.3,

$$\mathbf{w} = \text{hdet}(\epsilon^{d+1}\nu)\mathbf{w} = (\epsilon^{d+1}\nu)^{\otimes \ell}(\mathbf{w}).$$

Therefore

$$\mathbf{w} = ((-1)^{d+1}\nu)^{\otimes \ell}(\mathbf{w}) = (-1)^{(d+1)\ell}\nu^{\otimes \ell}(\mathbf{w}) = (-1)^{(d+1)\ell}\text{hdet}(\nu)\mathbf{w}.$$

Hence $\text{hdet}(\nu) = (-1)^{(d+1)\ell}$. But either d is odd or ℓ is even by Lemma 4.10 so $\text{hdet}(\nu) = (-1)^{(d+1)\ell} = 1$. \square

Since Theorem 4.11 does not need the noetherian hypothesis it might be reasonable to drop the noetherian assumption in [13, Conjecture 6.4].

Conjecture 4.12. *If ν is the Nakayama automorphism of an AS-regular algebra, then $\text{hdet}(\nu) = 1$.*

4.6. Let x_1, \dots, x_n be a basis for V , and $\mathbf{x}^t = (x_1, \dots, x_n)$. For every $\mathbf{w} \in V^{\otimes \ell}$, there exists a unique $n \times n$ matrix M whose entries are in $V^{\otimes \ell-2}$ such that $\mathbf{w} = \mathbf{x}^t M \mathbf{x}$ with the usual matrix multiplication. The $d = 3$ case of the following result is proved in [17, Corollary 9.3] and [4, Prop. 6.5].

Theorem 4.13. *Adopt the notation in §1.2. Choose a basis x_1, \dots, x_n for $V = S_1$ and define $\mathbf{x}^t = (x_1, \dots, x_n) \in V^n$. Let M be the unique $n \times n$ matrix with entries in $V^{\otimes \ell-2}$ such that $\mathbf{w} = \mathbf{x}^t M \mathbf{x}$. There is a unique matrix $Q \in \text{GL}(n, k)$ such that $(\mathbf{x}^t M)^t = Q M \mathbf{x}$. With respect to the ordered basis x_1, \dots, x_n , the Nakayama automorphism of S is*

$$\nu = (-1)^{d+1} Q^{-t} \in \text{GL}(n, k).$$

Thus, S is Calabi-Yau if and only if $Q = (-1)^{d+1} \text{Id}$.

Proof. By [1, Prop. 2.11], $(\mathbf{x}^t M)^t = Q M \mathbf{x}$ if and only if $(Q^{-t} \otimes \text{id}_V^{\otimes \ell-1})(\phi(\mathbf{w})) = \mathbf{w}$, so the result follows from Theorem 4.4. \square

Example 4.14. Let $S = k[x, y]$ be the polynomial algebra in two variables of degree 1. Since

$$\mathbf{w} = xy - yx = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

the matrix Q in Theorem 4.13 is $-\text{Id}$. Since S is Calabi-Yau, its Nakayama automorphism is the identity. Therefore

$$(\nu \otimes \text{id})\phi(\mathbf{w}) = (\text{id} \otimes \text{id})(yx - xy) = -\mathbf{w}$$

in accordance with the factor $(-1)^{d+1}$ that appears in Theorems 4.4 and 4.13 since $d = \text{gldim}(S) = 2$.

5. TWISTS

In this section, we examine the relation between twisting a superpotential and twisting an algebra by an automorphism à la Zhang [20].

5.1. Let $\sigma \in \text{GL}(V)$. Let $\mathbf{w} \in V^{\otimes \ell}$ and let W be a subspace of $V^{\otimes \ell}$. We define

$$\mathbf{w}^\sigma := (\sigma^{\ell-1} \otimes \cdots \otimes \sigma \otimes \text{id}_V)(\mathbf{w}).$$

and

$$W^\sigma := (\sigma^{\ell-1} \otimes \cdots \otimes \sigma \otimes \text{id}_V)(W) = \{\mathbf{w}^\sigma \mid \mathbf{w} \in W\}.$$

5.2. Let A be a graded algebra and $\sigma \in \text{Aut}(A)$. The Zhang twist of A by σ is the graded vector space $A^\sigma := A$ with multiplication $a * b := a\sigma^i(b)$ for $a \in A_i, b \in A$.⁴

Lemma 5.1. *Let R be a subspace of $V^{\otimes m}$ and let $A = TV/(R)$. If $\sigma \in \text{Aut}(A)$, then $A^\sigma \cong TV/(R^\sigma)$.*

Proposition 5.2. *Let $W \subseteq V^{\otimes \ell}$ be a subspace and $\sigma \in \text{GL}(V)$.*

- (1) $\partial^i(W^\sigma) = (\partial^i W)^\sigma$.
- (2) If $\mathcal{D}(W, i) = TV/(R)$ where $R \subseteq V^{\otimes \ell-i}$, then $\mathcal{D}(W^\sigma, i) = TV/(R^\sigma)$.
- (3) If $\sigma \in \text{Aut}(\mathcal{D}(W, i))$, then $\mathcal{D}(W^\sigma, i) \cong \mathcal{D}(W, i)^\sigma$.

Proof. (1) Let id denote the identity automorphism of V . Since

$$\begin{aligned} \partial(W^\sigma) &= \{(\psi \otimes \text{id}^{\otimes \ell-1})(\sigma^{\ell-1} \otimes \sigma^{\ell-2} \otimes \cdots \otimes \text{id})(\mathbf{w}) \mid \psi \in V^*, \mathbf{w} \in W\} \\ &= \{(\psi \sigma^{\ell-1} \otimes \sigma^{\ell-2} \otimes \cdots \otimes \text{id})(\mathbf{w}) \mid \psi \in V^*, \mathbf{w} \in W\} \\ &= \{(\psi \otimes \sigma^{\ell-2} \otimes \cdots \otimes \text{id})(\mathbf{w}) \mid \psi \in V^*, \mathbf{w} \in W\} \\ &= \{(\sigma^{\ell-2} \otimes \cdots \otimes \text{id})(\psi \otimes \text{id}^{\otimes \ell-1})(\mathbf{w}) \mid \psi \in V^*, \mathbf{w} \in W\} \\ &= (\partial W)^\sigma, \end{aligned}$$

an induction argument shows that $\partial^i(W^\sigma) = (\partial^i W)^\sigma$.

(2) If $\mathcal{D}(W, i) = TV/(R)$, then $R = \partial^i(W)$, so

$$\mathcal{D}(W^\sigma, i) = TV/(\partial^i(W^\sigma)) = TV/((\partial^i W)^\sigma) = TV/(R^\sigma)$$

by (1).

(3) If $\mathcal{D}(W, i) = TV/(R)$ and $\sigma \in \text{Aut}(\mathcal{D}(W, i))$, then

$$\mathcal{D}(W^\sigma, i) = TV/(R^\sigma) \cong (TV/(R))^\sigma = \mathcal{D}(W, i)^\sigma$$

by (2). □

⁴The paper [13] uses the convention $a * b = \sigma^j(a)b$ for $a \in A, b \in A_j$. The twist of A by $\sigma \in \text{Aut}(A)$ in the sense of [13] is the same as the twist of A by $\sigma^{-1} \in \text{Aut}(A)$ in our sense.

In the next result, the Nakayama automorphism of $TV/(R)$ is viewed as an element of $\text{GL}(V)$.

Proposition 5.3. *Adopt the notation in §1.2. If $\sigma \in \text{Aut}(S)$, then \mathbf{w}^σ is a $(-1)^{d+1}(\text{hdet}\sigma)^{-1}\sigma^\ell\nu$ -twisted superpotential.*

Proof. By Theorem 3.3, $\sigma^{\otimes\ell}(\mathbf{w}) = \text{hdet}(\sigma)\mathbf{w}$. By Theorem 4.4, $\phi(\mathbf{w}) = (-1)^{d+1}(\nu^{-1} \otimes \text{id}^{\otimes\ell-1})(\mathbf{w})$. Since

$$\begin{aligned} & ((-1)^{d+1}(\text{hdet}\sigma)^{-1}\sigma^\ell\nu \otimes \text{id}^{\otimes\ell-1})\phi(\mathbf{w}^\sigma) \\ &= (-1)^{d+1}(\text{hdet}\sigma)^{-1}(\sigma^\ell\nu \otimes \text{id}^{\otimes\ell-1})\phi((\sigma^{\ell-1} \otimes \cdots \otimes \sigma \otimes \text{id})(\mathbf{w})) \\ &= (-1)^{d+1}(\text{hdet}\sigma)^{-1}(\sigma^\ell\nu \otimes \text{id}^{\otimes\ell-1})(\text{id} \otimes \sigma^{\ell-1} \otimes \cdots \otimes \sigma)\phi(\mathbf{w}) \\ &= (-1)^{d+1}(\text{hdet}\sigma)^{-1}(\sigma^\ell\nu \otimes \sigma^{\ell-1} \otimes \cdots \otimes \sigma)(-1)^{d+1}(\nu^{-1} \otimes \text{id}^{\otimes\ell-1})(\mathbf{w}) \\ &= (\text{hdet}\sigma)^{-1}(\sigma^{\ell-1} \otimes \sigma^{\ell-2} \otimes \cdots \otimes \text{id})(\sigma \otimes \sigma \otimes \cdots \otimes \sigma)(\mathbf{w}) \\ &= (\text{hdet}\sigma)^{-1}(\sigma^{\ell-1} \otimes \sigma^{\ell-2} \otimes \cdots \otimes \text{id})((\text{hdet}\sigma)\mathbf{w}) \\ &= (\sigma^{\ell-1} \otimes \sigma^{\ell-2} \otimes \cdots \otimes \text{id})(\mathbf{w}) \\ &= \mathbf{w}^\sigma, \end{aligned}$$

\mathbf{w}^σ is a twisted superpotential for $(-1)^{d+1}(\text{hdet}\sigma)^{-1}\sigma^\ell\nu$. \square

In the next result, $\mathcal{D}(W, i)$ is of the form $TV/(R)$ so $\text{Aut}(\mathcal{D}(W, i)) \subseteq \text{GL}(V)$.

Proposition 5.4. *Let $W \subseteq V^{\otimes\ell}$. For all $i = 0, \dots, \ell - 1$, $\sigma \in \text{Aut}(\mathcal{D}(W^\sigma, i))$.*

Proof. Since $\mathcal{D}(W, i) := TV/(\partial^i W)$, $\sigma^{\otimes\ell-i}(\partial^i W) = \partial^i W$. Since

$$\begin{aligned} \sigma^{\otimes\ell-i}(\partial^i(W^\sigma)) &= \sigma^{\otimes\ell-i}((\partial^i W)^\sigma) \\ &= (\sigma \otimes \cdots \otimes \sigma)(\sigma^{\ell-i-1} \otimes \cdots \otimes \text{id})(\partial^i W) \\ &= (\sigma^{\ell-i-1} \otimes \cdots \otimes \text{id})(\sigma \otimes \cdots \otimes \sigma)(\partial^i W) \\ &= (\sigma^{\ell-i-1} \otimes \cdots \otimes \text{id})(\partial^i W) \\ &= (\partial^i W)^\sigma \\ &= \partial^i(W^\sigma), \end{aligned}$$

by Proposition 5.2 (1), σ extends to a graded algebra automorphism of $\mathcal{D}(W^\sigma, i) := TV/(\partial^i(W^\sigma))$. \square

The next result is a non-noetherian version of [13, Thm. 5.4(a),(b)].

Theorem 5.5. *Adopt the notation in §1.2. Let $\sigma \in \text{Aut}(S)$.*

- (1) S^σ is an m -Koszul AS-regular algebra of dimension d and Gorenstein parameter ℓ .
- (2) $\mathbf{w}_{S^\sigma} = (\mathbf{w}_S)^\sigma$.
- (3) $\sigma \in \text{Aut}(S^\sigma)$ and $\text{hdet}_{S^\sigma}(\sigma) = \text{hdet}_S(\sigma)$.
- (4) $\nu \in \text{Aut}(S^\sigma)$ and the Nakayama automorphism of S^σ is $\text{hdet}(\sigma)^{-1}\sigma^\ell\nu$.

Proof. (1) The minimal free resolution of k over S^σ is obtained by twisting the minimal free resolution of k over S by σ , so S^σ is also an m -Koszul AS-regular algebra of dimension d and Gorenstein parameter ℓ (cf. [6, Prop. 13]).

(2) Let $\mathbf{w} = \mathbf{w}_S$. Since $S = \mathcal{D}(\mathbf{w}, \ell - m)$, $S^\sigma = \mathcal{D}(\mathbf{w}, \ell - m)^\sigma$. But $\mathcal{D}(\mathbf{w}, \ell - m)^\sigma \cong \mathcal{D}(\mathbf{w}^\sigma, \ell - m)$ by Proposition 5.2(3) so, by Proposition 2.12, $\mathbf{w}_{S^\sigma} = \mathbf{w}^\sigma$.

(3) By Proposition 5.4, σ is a graded k -algebra automorphism of $\mathcal{D}(\mathbf{w}^\sigma, \ell - m)$. Hence $\sigma \in \text{Aut}(S^\sigma)$.

By Theorem 3.3, $\sigma^{\otimes \ell}(\mathbf{w}) = \text{hdet}_S(\sigma)\mathbf{w}$, so

$$\begin{aligned} \sigma^{\otimes \ell}(\mathbf{w}_{S^\sigma}) &= \sigma^{\otimes \ell}(\mathbf{w}^\sigma) \\ &= \sigma^{\otimes \ell}(\sigma^{\ell-1} \otimes \cdots \otimes \text{id})(\mathbf{w}) \\ &= (\sigma^{\ell-1} \otimes \cdots \otimes \text{id})\sigma^{\otimes \ell}(\mathbf{w}) \\ &= (\sigma^{\ell-1} \otimes \cdots \otimes \text{id})(\text{hdet}_S(\sigma)\mathbf{w}) \\ &= \text{hdet}_S(\sigma)\mathbf{w}^\sigma \\ &= \text{hdet}_S(\sigma)\mathbf{w}_{S^\sigma} \end{aligned}$$

by (2), so $\text{hdet}_{S^\sigma}(\sigma) = \text{hdet}_S(\sigma)$ by Theorem 3.3.

(4) Since $\nu\sigma = \sigma\nu$ by Corollary 4.9, and $\text{hdet}(\nu) = 1$ by Theorem 4.11,

$$\begin{aligned} \nu^{\otimes \ell}(\mathbf{w}_{S^\sigma}) &= \nu^{\otimes \ell}(\mathbf{w}^\sigma) \\ &= \nu^{\otimes \ell}(\sigma^{\ell-1} \otimes \cdots \otimes \text{id})(\mathbf{w}) \\ &= (\sigma^{\ell-1} \otimes \cdots \otimes \text{id})(\nu^{\otimes \ell})(\mathbf{w}) \\ &= (\sigma^{\ell-1} \otimes \cdots \otimes \text{id})(\text{hdet}(\nu)\mathbf{w}) \\ &= (\sigma^{\ell-1} \otimes \cdots \otimes \text{id})(\mathbf{w}) \\ &= \mathbf{w}^\sigma \\ &= \mathbf{w}_{S^\sigma} \end{aligned}$$

by (2) and Theorem 3.3. Thus, by Theorem 3.2, $\nu \in \text{Aut}(S^\sigma)$. By Theorem 4.4 and Proposition 5.3, $\nu_{S^\sigma} = (\text{hdet}(\sigma))^{-1}\sigma^\ell\nu$ is the Nakayama automorphism of S^σ . \square

Corollary 5.6. *Adopt the notation in §1.2 and suppose, further, that S is Calabi-Yau. If $\sigma \in \text{Aut}(S)$, then S^σ is Calabi-Yau if and only if $\sigma^\ell = \text{hdet}(\sigma)\text{id}_S$.*

Proof. Since $\nu = \text{id}_S$, S^σ is Calabi-Yau if and only if $\text{hdet}(\sigma)^{-1}\sigma^\ell = \nu_{S^\sigma} = \text{id}_{S^\sigma}$ by Theorem 5.5 if and only if $\sigma^\ell = \text{hdet}(\sigma)\text{id}_S$. \square

Proposition 5.7. *Let S be the polynomial ring on three indeterminates. Then S^σ is Calabi-Yau if and only if*

$$S^\sigma \cong \frac{k\langle x, y, z \rangle}{(yz - \xi zy, zx - \xi xz, xy - \xi yx)}$$

for some $\xi \in k$ such that $\xi^3 = 1$.

Proof. Suppose σ is in Jordan normal form with respect to the basis $\{x, y, z\}$ for $S_1 = V$. Since S is Calabi-Yau, S^σ is Calabi-Yau if and only if $\sigma^3 = \text{hdet}(\sigma)\text{id}_V = \det(\sigma)\text{id}_V$. Clearly, $\sigma^3 = \det(\sigma)\text{id}_V$ if and only if $\sigma = \text{diag}(\alpha, \alpha\xi, \alpha\xi^2)$ for some $\alpha, \xi \in k - \{0\}$ such that $\xi^3 = 1$. For such a σ , S^σ is as claimed. \square

6. JACOBIAN ALGEBRAS

Throughout §6, we assume that the characteristic of k does not divide ℓ .

Define $c : V^{\otimes \ell} \rightarrow V^{\otimes \ell}$ by

$$c(\mathbf{w}) := \frac{1}{\ell} \sum_{i=0}^{\ell-1} \phi^i(\mathbf{w}).$$

The i -th order Jacobian algebra of \mathbf{w} is $J(\mathbf{w}, i) := \mathcal{D}(c(\mathbf{w}), i)$

Lemma 6.1. *The set of superpotentials in $V^{\otimes \ell}$ is equal to the image of c , i.e., $\text{Im}(c) = \{\mathbf{w} \in V^{\otimes \ell} \mid \phi(\mathbf{w}) = \mathbf{w}\}$.*

Proof. If $\phi(\mathbf{w}) = \mathbf{w}$, then

$$c(\mathbf{w}) = \frac{1}{\ell} \sum_{i=0}^{\ell-1} \phi^i(\mathbf{w}) = \frac{1}{\ell} \sum_{i=0}^{\ell-1} \mathbf{w} = \mathbf{w},$$

so $\mathbf{w} = c(\mathbf{w}) \in \text{Im}(c)$. For the converse, since $\phi^\ell(\mathbf{w}) = \mathbf{w}$, if $c(\mathbf{w}) \in \text{Im}(c)$, then

$$\phi(c(\mathbf{w})) = \phi\left(\frac{1}{\ell} \sum_{i=0}^{\ell-1} \phi^i(\mathbf{w})\right) = \frac{1}{\ell} \sum_{i=0}^{\ell-1} \phi^{i+1}(\mathbf{w}) = c(\mathbf{w}).$$

□

Corollary 6.2. *Let $0 \neq \mathbf{w} \in V^{\otimes \ell}$. If $S = J(\mathbf{w}, \ell - m)$ is an m -Koszul Calabi-Yau algebra, then $\mathbf{w}_S = c(\mathbf{w})$ up to a non-zero scalar multiple.*

Proof. If $S = J(\mathbf{w}, \ell - m) := \mathcal{D}(c(\mathbf{w}), \ell - m)$ is Calabi-Yau, then it is AS-regular by Lemma 2.3. Since $c(\mathbf{w})$ is a (twisted) superpotential by Lemma 6.1, $c(\mathbf{w}) = \mathbf{w}_S$ up to a non-zero scalar multiple by Proposition 2.12. □

Theorem 6.3. *Adopt the notation in §1.2. If d is odd, then the following are equivalent:*

- (1) S is Calabi-Yau.
- (2) $\phi(\mathbf{w}) = \mathbf{w}$.
- (3) $c(\mathbf{w}) = \mathbf{w}$.
- (4) $S \cong J(\mathbf{w}, \ell - m)$.

Proof. (1) \Leftrightarrow (2). By Corollary 4.5, S is Calabi-Yau if and only if $\phi(\mathbf{w}) = (-1)^{d+1}\mathbf{w}$. Since d is odd, S is Calabi-Yau if and only if $\phi(\mathbf{w}) = \mathbf{w}$

(2) \Leftrightarrow (3). If $\phi(\mathbf{w}) = \mathbf{w}$, then $c(\mathbf{w}) = \mathbf{w}$. Conversely, if $c(\mathbf{w}) = \mathbf{w}$, then $\phi(\mathbf{w}) = \mathbf{w}$ by Lemma 6.1.

(3) \Leftrightarrow (4). If $c(\mathbf{w}) = \mathbf{w}$, then $S = \mathcal{D}(\mathbf{w}, \ell - m) = \mathcal{D}(c(\mathbf{w}), \ell - m) =: J(\mathbf{w}, \ell - m)$ by Proposition 2.12. Conversely, since S is m -Koszul AS-regular and $c(\mathbf{w})$ is a (twisted) superpotential by Lemma 6.1, if $S = J(\mathbf{w}, \ell - m) := \mathcal{D}(c(\mathbf{w}), \ell - m)$, then $\mathbf{w} = c(\mathbf{w})$ by Proposition 2.12 □

Theorem 6.4. *Adopt the notation in §1.2. If d is even, then the following are equivalent:*

- (1) S is Calabi-Yau.
- (2) $\phi(\mathbf{w}) = -\mathbf{w}$.
- (3) $\tilde{c}(\mathbf{w}) = \mathbf{w}$.
- (4) $S \cong \tilde{J}(\mathbf{w}, \ell - m)$.

where $\tilde{c}(\mathbf{w}) := \frac{1}{\ell} \sum_{i=0}^{\ell-1} (-1)^i \phi^i(\mathbf{w})$ and $\tilde{J}(\mathbf{w}, i) := \mathcal{D}(\tilde{c}(\mathbf{w}), i)$.

Proof. (1) \Leftrightarrow (2). By Corollary 4.5, S is Calabi-Yau if and only if $\phi(\mathbf{w}) = (-1)^{d+1}\mathbf{w}$. Since d is even, S is Calabi-Yau if and only if $\phi(\mathbf{w}) = -\mathbf{w}$

(2) \Rightarrow (3). If $\phi(\mathbf{w}) = -\mathbf{w}$, then

$$\tilde{c}(\mathbf{w}) = \frac{1}{\ell} \sum_{i=0}^{\ell-1} (-1)^i \phi^i(\mathbf{w}) = \frac{1}{\ell} \sum_{i=0}^{\ell-1} (-1)^i (-1)^i \mathbf{w} = \mathbf{w}.$$

(3) \Rightarrow (2). Suppose $\tilde{c}(\mathbf{w}) = \mathbf{w}$. Since d is even, Lemma 4.10 tells us that S is Koszul and $\ell = d$. Therefore $(-1)^\ell \phi^\ell(\mathbf{w}) = \mathbf{w}$. Hence

$$\begin{aligned} \phi(\mathbf{w}) &= \phi(\tilde{c}(\mathbf{w})) = \phi\left(\frac{1}{\ell} \sum_{i=0}^{\ell-1} (-1)^i \phi^i(\mathbf{w})\right) \\ &= \frac{1}{\ell} \sum_{i=0}^{\ell-1} (-1)^i \phi^{i+1}(\mathbf{w}) \\ &= -\frac{1}{\ell} \sum_{i=0}^{\ell-1} (-1)^{i+1} \phi^{i+1}(\mathbf{w}) \\ &= -\frac{1}{\ell} \sum_{i=0}^{\ell-1} (-1)^i \phi^i(\mathbf{w}) \\ &= -\tilde{c}(\mathbf{w}) = -\mathbf{w}. \end{aligned}$$

(3) \Leftrightarrow (4). If $\tilde{c}(\mathbf{w}) = \mathbf{w}$, then $S = \mathcal{D}(\mathbf{w}, \ell - m) = \mathcal{D}(\tilde{c}(\mathbf{w}), \ell - m) =: \tilde{J}(\mathbf{w}, \ell - m)$ by Proposition 2.12. For the converse, since $\phi(\tilde{c}(\mathbf{w})) = -\tilde{c}(\mathbf{w})$ by the above proof, $\tilde{c}(\mathbf{w})$ is a twisted superpotential. Since S is m -Koszul AS-regular, if $S = \tilde{J}(\mathbf{w}, \ell - m) := \mathcal{D}(\tilde{c}(\mathbf{w}), \ell - m)$, then $\mathbf{w} = \tilde{c}(\mathbf{w})$ by Proposition 2.12 \square

7. AS-REGULAR ALGEBRAS OF DIMENSIONS 2 AND 3

Let $S = TV/(R)$ be an AS-regular algebra. Frequently, $\text{hdet}(\sigma) = \det(\sigma)$ for all $\sigma \in \text{Aut}(S)$. In this section we examine some (non-)examples of this phenomenon.

7.1. Let S be a noetherian AS-regular algebra of dimension 2 generated in degree 1. Then S is Koszul and isomorphic to $TV/(\mathbf{w})$ for some rank-two tensor $\mathbf{w} \in V^{\otimes 2}$. By [12, Lem. 4.2], for example, there exists $\sigma \in \text{Aut}(S)$ such that $\text{hdet}(\sigma) \neq \det(\sigma)$ if and only if $S \cong k\langle x, y \rangle / (xy + yx)$ or, equivalently, if and only if $\mathbf{w} \in \text{Sym}^2 V$.

We will now show that something similar happens for the 3-dimensional case.

7.2. Let $S = TV/(R)$ be a noetherian 2-Koszul AS-regular algebra of dimension 3. Then $\dim V = 3$.

The symmetric group, \mathfrak{S}_3 , acts on $V^{\otimes 3}$ by

$$\theta(v_1 \otimes v_2 \otimes v_3) := v_{\theta(1)} \otimes v_{\theta(2)} \otimes v_{\theta(3)}, \quad \theta \in \mathfrak{S}_3.$$

Let $s, a : V^{\otimes 3} \rightarrow V^{\otimes 3}$ be the linear maps

$$\begin{aligned} s(\mathbf{w}) &:= \frac{1}{|\mathfrak{S}_3|} \sum_{\theta \in \mathfrak{S}_3} \theta(\mathbf{w}) \\ a(\mathbf{w}) &:= \frac{1}{|\mathfrak{S}_3|} \sum_{\theta \in \mathfrak{S}_3} (\text{sgn } \theta) \theta(\mathbf{w}), \end{aligned}$$

and define the following subspaces of $V^{\otimes 3}$:

$$\begin{aligned} \text{Sym}^3 V &:= \{\mathbf{w} \in V^{\otimes 3} \mid \theta(\mathbf{w}) = \mathbf{w} \text{ for all } \theta \in \mathfrak{S}_3\}, \\ \text{Alt}^3 V &:= \{\mathbf{w} \in V^{\otimes 3} \mid \theta(\mathbf{w}) = (\text{sgn } \theta) \mathbf{w} \text{ for all } \theta \in \mathfrak{S}_3\}. \end{aligned}$$

Fix a basis \mathbf{w}_0 for $\text{Alt}^3(V) = k\mathbf{w}_0$ and let

$$\mu : V^{\otimes 3} \rightarrow k$$

be the unique map such that $c(\mathbf{w}) = s(\mathbf{w}) + \mu(\mathbf{w})\mathbf{w}_0$. The map μ plays a key role in [11].

Theorem 7.1. *Let $S = TV/(R)$ be a noetherian 2-Koszul AS-regular algebra of dimension 3. If $\text{hdet}(\sigma) \neq \det(\sigma)$ for some $\sigma \in \text{Aut}(S)$, then $c(\mathbf{w}_S) \in \text{Sym}^3 V$.*

Proof. Let $\mathbf{w} = \mathbf{w}_S$. Although μ depends on the choice of \mathbf{w}_0 , it is an easy exercise to show that (or, essentially by the definition of the determinant) $\mu(\sigma^{\otimes 3}(\mathbf{v})) = (\det \sigma)\mu(\mathbf{v})$ for all $\sigma \in \text{GL}(V)$ and all $\mathbf{v} \in V^{\otimes 3}$.

By Theorem 3.3, if $\sigma \in \text{Aut}(S)$, then $\sigma^{\otimes 3}(\mathbf{w}) = \text{hdet}(\sigma)\mathbf{w}$ so

$$\det(\sigma)\mu(\mathbf{w}) = \mu(\sigma^{\otimes 3}(\mathbf{w})) = \mu(\text{hdet}(\sigma)\mathbf{w}) = \text{hdet}(\sigma)\mu(\mathbf{w}).$$

If $\text{hdet}(\sigma) \neq \det(\sigma)$, then $\mu(\mathbf{w}) = 0$, so $c(\mathbf{w}) = \mu(\mathbf{w})\mathbf{w}_0 + s(\mathbf{w}) = s(\mathbf{w}) \in \text{Sym}^3 V$. \square

For the rest of the paper, we write $\mathcal{D}(\mathbf{w}) := \mathcal{D}(\mathbf{w}, 1)$ and $J(\mathbf{w}) := J(\mathbf{w}, 1)$.

Theorem 7.2. *Let V be a 3-dimensional vector space and $0 \neq \mathbf{w} \in V^{\otimes 3}$ such that $J(\mathbf{w})$ is Calabi-Yau. Then the following are equivalent:*

- (1) $\text{hdet}(\sigma) \neq \det(\sigma)$ for some $\sigma \in \text{Aut}(J(\mathbf{w}))$;
- (2) $c(\mathbf{w}) \in \text{Sym}^3 V$;
- (3) $R \subseteq \text{Sym}^2 V$.

Proof. Let $S = J(\mathbf{w})$. By Corollary 6.2, $\mathbf{w}_S = c(\mathbf{w})$.

(1) \Rightarrow (2) By Theorem 7.1, $c(\mathbf{w}) = c(c(\mathbf{w})) = c(\mathbf{w}_S) \in \text{Sym}^3 V$.

(2) \Rightarrow (1) Since $c(\mathbf{w}) = s(\mathbf{w}) + a(\mathbf{w}) \in \text{Sym}^3 V \oplus \text{Alt}^3 V$, $c(\mathbf{w}) \in \text{Sym}^3 V$ if and only if $c(\mathbf{w}) = s(\mathbf{w})$. So, if $c(\mathbf{w}) \in \text{Sym}^3 V$ there is a basis x, y, z for V such that

$$c(\mathbf{w}) = xyz + yzx + zxy + xzy + yxz + zyx + \alpha x^3 + \beta y^3 + \gamma z^3$$

and $(\alpha, \beta, \gamma) \in \{(1, 0, 0), (1, 1, 0), (\alpha, \alpha, \alpha) \mid \alpha \in k\}$ by [11, §1.8.4]. Let

$$\sigma = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \text{if } (\alpha, \beta, \gamma) \text{ is } (1, 0, 0) \text{ or } (\alpha, \alpha, \alpha) \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } (\alpha, \beta, \gamma) = (1, 1, 0). \end{cases}$$

In all cases $\sigma^{\otimes 3}(\mathbf{w}) = \mathbf{w}$ so, by Theorem 3.2, σ extends to an automorphism of $J(\mathbf{w})$ and, by Theorem 3.3, $\text{hdet}(\sigma) = 1 \neq -1 = \det(\sigma)$.

(2) \Leftrightarrow (3) This is the content of [11, §3.1]. \square

Remark 7.3. Let V be a 3-dimensional vector space and $0 \neq \mathbf{w} \in V^{\otimes 3}$. By [11, Thm. 3.2], $c(\mathbf{w}) \notin \text{Sym}^3 V$ if and only if $J(\mathbf{w})$ is a deformation quantization of the polynomial ring $k[x, y, z]$. By Theorem 7.2, if $J(\mathbf{w})$ is Calabi-Yau and a deformation quantization of $k[x, y, z]$, then $\text{hdet}(\sigma) = \det(\sigma)$ for every $\sigma \in \text{Aut } J(\mathbf{w})$.

The point of our final example is to show that the Zhang twists of a single noetherian Koszul Calabi-Yau algebra of dimension 3 can behave very differently from one another.

Example 7.4. Let $V = kx \oplus ky \oplus kz$ and $\mathbf{w} = xyz + yzx + zxy + xzy + yxz + zyx \in \text{Sym}^3 V$. Then

$$S = \mathcal{D}(\mathbf{w}) = J(\mathbf{w}) = \frac{k\langle x, y, z \rangle}{(yz + zy, zx + xz, xy + yx)}.$$

is a noetherian Koszul Calabi-Yau algebra of dimension 3. The linear maps $\sigma_i : V \rightarrow V$ defined by

$$\sigma_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \sigma_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

satisfy $\sigma_i^{\otimes 3}(\mathbf{w}) = \mathbf{w}$ so $\sigma_1, \sigma_2 \in \text{Aut}(S)$ and, by Theorem 3.3, $\text{hdet}(\sigma_i) = 1$.

Since $\sigma_2^3 = \text{id}$ and $\sigma_1^3 \neq \text{id}$, it follows from Corollary 5.6 that S^{σ_2} is Calabi-Yau but S^{σ_1} is not.

A calculation shows that

$$\begin{aligned} \mathbf{w}^{\sigma_1} &= xz^2 + y^2x + zxy + xy^2 + yxz + z^2x \quad \text{and} \\ \mathbf{w}^{\sigma_2} &= z^3 + x^3 + y^3 + zxy + xyz + yzx \end{aligned}$$

so $c(\mathbf{w}^{\sigma_1}) \in \text{Sym}^3 V$ but $c(\mathbf{w}^{\sigma_2}) \notin \text{Sym}^3 V$.

Since $c(\mathbf{w}^{\sigma_2}) \notin \text{Sym}^3 V$, Theorem 7.1 says that $\text{hdet}(\tau) = \det(\tau)$ for all $\tau \in \text{Aut}(S^{\sigma_2})$. On the other hand, by Theorem 5.5, $\sigma_1 \in \text{Aut}(S^{\sigma_1})$ and $\text{hdet}_{S^{\sigma_1}}(\sigma_1) = \text{hdet}_S(\sigma_1) = 1 \neq -1 = \det(\sigma_1)$.

By Theorem 5.5(2),

$$S^{\sigma_1} \cong \mathcal{D}(\mathbf{w}^{\sigma_1}) \cong \frac{k\langle x, y, z \rangle}{(z^2 + y^2, yx + xz, xy + zx)}$$

and

$$S^{\sigma_2} \cong \mathcal{D}(\mathbf{w}^{\sigma_2}) \cong \frac{k\langle x, y, z \rangle}{(yz + x^2, zx + y^2, xy + z^2)}.$$

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